Around Newton's Theorem on the Gravitational Attraction Induced by Ellipsoids and Null Quadrature Domains

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Issues to Be Discussed

- Newton's theorem and the connection to null quadrature domains
- Generalized Newtonian potential
- Results and a conjecture
- Tools of the proof
- Phragmén-Lindelöf type theorems

1 Newton's theorem and the connection to null quadrature domains

Suppose Ω is a body in $\mathbb{R}^3$ with mass density ρ, then the gravitational attraction at a point x is given by

$$\mathbf{F}(x) = -\int_\Omega \frac{x-y}{|x-y|^3}\rho(y)dy. \tag{1}$$

The gravitational potential shall be determined by the *Newtonian potential*

$$V\rho(x) = \int_\Omega \frac{\rho(y)}{|x-y|}dy. \tag{2}$$

Here we assume $\rho \equiv const.$, so we let $\rho \equiv 1$.

Consider a *homoeoiodal sphere* $\lambda B \setminus B$ $(\lambda > 1)$. In this case one may easily compute the force $\mathbf{F}$ in (1) and deduce that the gravitational attraction at any internal points is zero:

$$\mathbf{F}(x) = -\int_{\lambda B\setminus B} \frac{x-y}{|x-y|^3}dy = 0, \qquad x \in B. \tag{3}$$

A less trivial is Newton's theorem. Let E be an ellipsoid centered at the origin. We call $\lambda E \setminus E$ a *homoeoiodal ellipsoid*

Theorem (Newton): *The gravitational attraction at any internal point of homoeoiodal ellipsoid is* zero.

No Gravity in the Cavity

Question:

Are ellipsoids the only bodies having the property that gravitational force induced by a homoeoid is zero at all internal points?

So suppose the **homoeoid** $\lambda K \setminus K$ $(\lambda > 1)$ produces no gravity force in the cavity K, then

$$\int_{\lambda K \setminus K} \frac{1}{|x-y|} dy = \text{const.} \quad \text{for all } x \in K. \tag{4}$$

Taking the Taylor expansion of the potential

$$\int_K \frac{1}{|x-y|} dy = \sum_{m=0}^{\infty} P_m(x),$$

(P_m is a homogeneous polynomial of degree m) and insert it in (4), we get

$$\int_{\lambda K \setminus K} \frac{1}{|x-y|} dy = \sum_{m=0}^{\infty} \left(\lambda^{2-m} - 1\right) P_m(x) = \text{const.}$$

The gravitational force induces by the homoeoid $\lambda K \setminus K$ attracts no internal points, for some $\lambda > 1$ $\Leftrightarrow$ $\int_K \frac{dy}{|x-y|} =$ quadratic poly. **for** $x \in K$

Historical Note:

1. **Calculation of the potential of ellipsoids:** The problem of determining the gravitational potential of an ellipsoid was one of the most important problems in mathematics for more than two centuries after Newton had enunciated his celebrated law of gravitation. Some of the most distinguished mathematicians of this time, Maclaurin, D'Alembert, Legendre, Laplace, Gauss, Jacobi, Poisson, Dirichlet, Cayley,..., were involoved in this problem.

2. **Application to the shape of earth:** Newton addressed the question of the shape of the earth (or other planet). He knew that the shape would be spherical in the absence of rotation, and wanted to predict the departure from the spherical shape due to the centrifugal acceleration of the earth's rotation.

   He made several assumptions:

   - the planet is a fluid mass of uniform density;
   - the rotation is that of a rigid body;
   - the shape is that of an oblate spheroid whose minor axis is the axis of rotation: $\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2}, a > b$.

   He measured the *ellipticity* by $\varepsilon = \frac{a-b}{a}$ and got $\varepsilon \sim 1/230$. We know now that the actual ellipticity of the earth is $\sim 1/294$; and this discrepancy is interpreted in terms of the inhomogeneity of the earth.

   Later Maclaurin showed an oblate spheroid could be a steady-state figure and actually his reasoning were capable of generalization to ellipsoids in which no two axes are equal. That generalization required the exact expression of the potential of an homogeneous ellipsoid.Therefore this computation attracted the attention of of the eminent mathematicians of the eighteenth century.

   S. Chandrasekhar, *Ellipsoidal figures of equilibrium,* (1969).

   N.R. Libovitz, *The mathematical development of the classical ellipsoids,* 1998.

3. **Characterization of the potential of ellipsoids:**
The question

$$\boxed{\begin{array}{c}\int_K \frac{dy}{|x-y|} = \text{quadratic poly.} \\ \text{for } x \in K\end{array}} \quad \Leftrightarrow \quad \boxed{\text{K } \textbf{is an ellipsoid}}$$

was addressed by Hölder (1932) in $\mathbb{R}^2$, Dive (1931) and Nikliborc (1932) in $\mathbb{R}^3$. Dive and Nikliborc gave an affirmative answer to the above question. Later on, analogous results were proved in arbitrary dimension by DiBenedetto and Friedman in 1986 and by Karp in 1995

1.1 Connection to Variational Inequalities

A variational inequality may be written:

$$\Delta u \leq f, \quad u \geq 0, \quad (\Delta u - f)u = 0 \tag{5}$$

in a domain G of $\mathbb{R}^n$ and the non-coincidence set is $\Omega := \{x \in G : \; u(x) > 0\}$. Now, if

$$\boxed{\int_K \frac{dy}{|x-y|} = q(x) = \text{quadratic poly. for } x \in K,}$$

then

$$u(x) = \frac{-1}{4\pi}\left(q(x) - \int_K \frac{dy}{|x-y|}\right) \in C^1(\mathbb{R}^3)$$

then u is a solution to a global variational inequality:

$$\boxed{\Delta u \leq 1, \quad u \geq 0, \quad (\Delta u - 1)u = 0, \qquad (6)}$$

where the non-coincidence set in this case is $\Omega := \mathbb{R}^3 \backslash K$. **Thus u is a global solution of the variational inequality.**

The system (6) means that $u = |\nabla u| = 0$ on $K = \mathbb{R}^3 \setminus \Omega$, so by applying Green's identity with u above and a harmonic function h in Ω, we have

$$\int_\Omega h dx = \int_\Omega h \Delta u dx = \int_{\partial\Omega} \left(h\frac{\partial u}{\partial \nu} - u\frac{\partial h}{\partial \nu}\right) dS = 0$$

Definition 1 *An open set $\Omega \subset \mathbb{R}^n$ is called a* **null quadrature domain (QD)** *if*

$$\int_\Omega h dx = 0$$

for all harmonic and integrable functions h in Ω.

Examples of null quadrature domains - global solutions of the variational inequlity:

1. Exterior of an ellipsoid $\mathbb{R}^n \setminus E$;
2. Half-space $\{x_n < a\}$;
3. The complement of a slab $\{x_n < a\} \cup \{x_n > b\}$.
4. The complement of a cylinder over an ellipsoid $\{(\frac{x_1}{a})^2 + (\frac{x_2}{b})^2 > 1 : \; x_3 \in \mathbb{R}\}$.

Remark 2 *Our aim is the characterization of null quadrature domains in $\mathbb{R}^n$, or alternatively, global solutions of a variational inequality, <u>with unbounded complement.</u>*

We see that the characterization relies the computations of potentials. Therefore we need a tool which will allow us to compute a "potential" of unbounded bodies.

2 Generalized Newtonian Potentials

2.1 Definitions

The generalized Newtonian potential was introduced and studied Margulis and Karp. Here we repeat its definition.

The **Newtonian potential of a measure μ** with having a compact support is:

$$V\mu(x) = \frac{1}{(n-2)\omega_n} \int \frac{d\mu(y)}{|x-y|^{n-2}}, \qquad n \geq 3$$

It satisfies the equation

$$\Delta V\mu = -\mu \tag{7}$$

Our idea is to define the generalized Newtonian potentials by means of Poisson equation (7).

Definition 3

1. *Let $\mathcal{L}$ be the Banach space of all Radon measures such that*

$$\|\mu\|_{\mathcal{L}} := \int \frac{d|\mu|(x)}{(1+|x|)^{n+1}} \tag{8}$$

is finite. Note that $\mathcal{L}$ contains all measures of the type: $d\mu = f dx, f \in L^\infty(\mathbb{R}^n)$.

2. *For $\mu \in \mathcal{L}$ and a multi-index α with $|\alpha| = 3$ we define the third order formal derivative of $V\mu$ by*

$$\langle V^\alpha(\mu), \varphi\rangle = -\int \partial^\alpha(V\varphi)(x)d\mu(x), \qquad \varphi \in \mathcal{S}, \tag{9}$$

where $\mathcal{S}$ is the Schwartz class of rapidly decreasing functions. The operator $V^\alpha : \mathcal{L} \to \mathcal{S}'$ is continuous ([KM]).

3. *The* **generalized Newtonian potential** *of a measure μ in $\mathcal{L}$ is denoted by $V[\mu]$ and is the set of all solutions to the system*

$$\begin{cases} \Delta v = -\mu \\ \partial^\alpha v = V^\alpha(\mu), \qquad |\alpha| = 3. \end{cases} \tag{10}$$

The existence of solutions to (10) were proved in [KM], furthermore, two elements of $V[\mu]$ defer by a harmonic polynomial of degree less or equal to two.

2.2 Examples of generalized Newtonian potential:

We let $\mathcal{H}_2$ denote the space of harmonic polynomials of degree ≤ 2.

1. **A half-space:** Let $\mu = \chi_\Omega$, where $\Omega = \{x_n < a\}$, then $u \in V[\chi_\Omega] \Leftrightarrow$

$$u(x) = \left\{ \begin{array}{cc} 0, & x_n \geq a \\ \frac{(x_n-a)^2}{2}, & x_n < a \end{array} \right\} + h_2(x), \qquad h_2 \in \mathcal{H}_2;$$

2. **A cone with vertex at the origin:** Let $\mu = \chi_K$, where K is a cone, then $u \in V[\chi_K] \Leftrightarrow$

$$u(x) = h_2(x) \log |x| + |x|^2 \phi(\frac{x}{|x|}) + \mathbf{a} \cdot x + b, \qquad (11)$$

where $h_2 \in \mathcal{H}_2$, h_2 homogeneous and $\phi \in C^1(\mathbb{S}^{n-1})$. The log term reflects the singularity of the boundary both at the origin and at infinity. If the cone K is a half-space, then in (11) $h_2 \equiv 0$, however, there are other cones with $h_2 \equiv 0$.

2.3 The generalized Newtonian potential of the complement of null QD

By means of the third order derivatives of the generalized potential, we get that if $\Omega \subset \mathbb{R}^n$ is a null QD and $\mathrm{supp}(\varphi) \subset \mathbb{R}^n \setminus \Omega$, then the Newton potential $V\varphi$ is harmonic in Ω which leads to

$$V^\alpha(\chi_{\mathbb{R}^n\setminus\Omega})(x) = 0, \quad \text{for all multi-indices} \alpha, |\alpha| = 3, x \in \mathbb{R}^n\setminus\Omega.$$

Theorem 4 *An open set Ω in $\mathbb{R}^n$ is a null quadrature domain if and only if there is $v \in V[\chi_{\mathbb{R}^n\setminus\Omega}]$ such that v coincides with a quadratic polynomial in $\mathbb{R}^n \setminus \Omega$.*

Compare with:

The gravitational force induces by the homoeoid $\lambda K \setminus K$ attracts no internal points, for some $\lambda > 1$	$\Leftrightarrow$	$\int_K \frac{dy}{\lvert x-y\rvert} =$ quadratic poly. **for** $x \in K$

Remark 5

(a) Theorem 4 suggests a generalization of Newton's theorem of no gravitational force in the interior of homoeoidal ellipsoids to unbounded sets.

(b) A different extension of this theorem to hyperbolic quadratic surfaces was done by Arnold and some of his colleagues.

3 Results and a Conjecture

The characterization of null quadrature domains in R^n ($n \geq 3$) has been an open problem throughout the past two and a half decades.

3.1 Characterization of null quadrature domains in $\mathbb{R}^2$:

Theorem 6 (Sakai, 1980) *The open set $\Omega \subset \mathbb{R}^2$ is a null quadrature domain if and only if Ω is one of the following:*

$\int_\Omega h dx = 0$ for h harmonic and integrable in Ω

1. *The exterior on an ellipse;*
2. *The complement of a strip;*
3. *A half-plane;*
4. *The non-convex domain bounded by a parabola.*

3.2 Conjecture and known results in $\mathbb{R}^n$:

Cojecture 7 (Karp & Margulis, 1996) *The open set $\Omega \subset \mathbb{R}^n$ is a null quadrature domain if and only if Ω is one of the following:*

1. *The exterior on an ellipsoid;*
2. *The complement of a slab;*
3. *A half-space;*
4. *The non-convex domain bounded by an elliptic paraboloid;*
5. *A cylinder over 1. or 4.*

(All these sets are indeed null QD)

Known results in higher dimensions:

(i) If $\mathbb{R}^n \setminus \Omega$ is bounded, then it is an ellipsoid (Friedman & Sakai, 1986);

(ii) If in a certain coordinates system $\partial\Omega \subset \{x : x_{n-1}^2 + x_n^2 < R^2\}$ for some positive R ($\partial\Omega \subset$ infinite cylinder), then $\mathbb{R}^n \setminus \Omega$ is an ellipsoid or a cylinder over an ellipsoid (Karp & Margulis, 1996).

(iii) If a domain Ω is bounded by hyperboloid, or hyperboloid paraboloid, then Ω is **NOT** a null QD. (Karp & Margulis, 1996).

Explanation of (iii):

Assume Ω is a null quadrature domain which is bounded by a hyperboloid and $0 \in \partial\Omega$. Let $v \in V[\chi_\Omega]$ (a generalized Newtonian potential of χ_Ω) such that $v(0) = 0$. Then

$$v_\epsilon(x) := \frac{v(\epsilon x)}{\epsilon^2} \in V[\chi_{(\epsilon\Omega)}],$$

where $\epsilon\Omega = \{x : \epsilon x \in \Omega\}$, moreover, $\epsilon\Omega$ is a null QD for each ϵ. Now, as $\epsilon \to \infty$, $\epsilon\Omega \to$ a cone K. By some properties of the g.N.p, $v_\epsilon \to v_K \in V[\chi_K]$. Now a cone K is null QD if and only if is a half-space. But any cone which is a limit of a domain bounded by hyperboloid, is NOT a half-space.

3.3 A new results:

The new result is an attempt to classify null QD whose boundary is contained in a slab $\{|x_n| \leq R\}$.

Theorem 8 (Karp) *If $\Omega \subset \mathbb{R}^n$ is a null quadrature domain such that in a certain coordinates system $\partial\Omega \subset \{x : |x_n| \leq R\}$ for some positive R and*

$$\liminf_{r\to\infty} \frac{\mathrm{Cap_n}\left((\mathbb{R}^\mathrm{n} \setminus \Omega) \cap \mathrm{B_r}\right)}{\mathrm{Cap_n}(\mathrm{B_r})} > 0, \tag{12}$$

then Ω is a half-space or a complement of a slab.

Here $\mathrm{Cap_n}(\mathrm{F})$ denotes the Newtonian capacity of a compact set F in $\mathbb{R}^n$ and B_r is a ball of radius r.

(12) is similar to *Lebegue density at infinity* but with $\mathrm{Cap_n}$ replacing the Lebseque measure.

4 Tools of the Proof

The main ideas of the proof are the following: Assume Ω is a Null QD and and let $v \in V[\chi_{\mathbb{R}^n \setminus \Omega}]$.

(a) We show that $v(x_1, ..., x_n)$ is independent of some of the variables.

(b) Step (a) implies that $\chi_{\mathbb{R}^n \setminus \Omega}$ is a cylindrical set and thus it reduces the problem to known cases or to a lower dimension.

(c) In order to obtain (a) we combine growth properties of the *generalized Newtonian potential* with Phragmén-Lindelöf type theorems.

(d) **Convexity:** In order to prove the new result we also need convexity of the complement. That is, assume u satisfies

$$\begin{cases} \Delta u = \chi_\Omega, & \text{in } \mathbb{R}^n \\ u = |\nabla u| = 0, & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{13}$$

then $\mathbb{R}^n \setminus \Omega$ is convex.

In case $u \geq 0$ this follows from Caffarelly, *Compactness methods in free boundary probelms*, 1980.

Without the restriction $u \geq 0$:

in Caffarelli, Karp & Shahgholian (2000) it was proved that any u satisfying (13) is non-negative.

4.1 Growth properties of the generalized Newtonian potential:

(i) **Lemma 9** *Suppose* $f \in L^\infty$ *and* $\operatorname{supp}(f) \subset \{x_{n-1}^2 + x_n^2 \le R^2\}$. *Then there is* $v \in V[f dx]$ *such that*

$$|v(x)| \le C\|f\|_{L^\infty} \log(2 + |x|)$$

and

$$|\nabla v(x)| \le C\|f\|_{L^\infty}.$$

(Karp & Margulis, 1996);

(ii) **Lemma 10** *Suppose* $f \in L^\infty$ *and* $\operatorname{supp}(f) \subset \{|x_n| \le R\}$. *Then there is* $v \in V[f dx]$ *such that*

$$|v(x)| \le C\|f\|_{L^\infty} C(1 + |x|) \log(2 + |x|)$$

and

$$|\nabla v(x)| \le C\|f\|_{L^\infty} C \log(2 + |x|).$$

(Karp, 2007).

5 Phragmén-Lindelöf type theorems

The principle idea of Phragmén-Lindelöf type theorems is that the maximum principle holds in unbounded domains provided that the function has a limited growth at infinity. The rate of growth depends upon the "size" of the domain in question. As the "size" increases the rate of growth decreases. We give few example which emphasize this phenomena.

Example 1: Let $\Omega \subseteq \{z \in \mathbb{C} : |\arg(z)| < \frac{\pi}{2\alpha}\}$. Suppose u is subharmonic in Ω, continuous in $\overline{\Omega}$ and $u \leq 0$ on the boundary $\partial\Omega$. Then either $u \leq 0$ in Ω or

$$\liminf_{r\to\infty} \frac{\sup_{\{\Omega\cap\{|x|=r\}\}} |u(x)|}{r^\alpha} > 0.$$

The growth here is sharp: take $u(z) = r^\alpha \cos(\alpha\theta)$, then u vanishes on $\partial\Omega$ and positive in Ω.

Example 2: We take the n dimensional version of Example 1. Let $\Omega \subseteq \{x \in \mathbb{R}^n : x_n > |x| \cos\alpha\}$. Let $\Sigma = \mathbb{S}^{n-1} \cap \Omega$ and λ be the first eigenvalue of $-\Delta_{\mathbb{S}^{n-1}}$ with vanishing data on $\partial\Sigma$. Taking β the positive rote of $\beta^2 + (n-2)\beta = \lambda$, we get: Suppose u is subharmonic in Ω, continuous in $\overline{\Omega}$ and $u \leq 0$ on the boundary $\partial\Omega$. Then either $u \leq 0$ in Ω or

$$\liminf_{r\to\infty} \frac{\sup_{\{\Omega\cap\{|x|=r\}\}} |u(x)|}{r^\beta} > 0.$$

New Result: Let $\mathrm{Cap_n(F)}$ denote the n dimensional Newtonian capacity.

Theorem 11 (Karp) *Let u be a subharmonic function in an unbounded domain Ω, continuous on $\overline{\Omega}$ and such that $u \leq 0$ on $\partial\Omega$. Suppose*

$$\liminf_{r\to\infty} \frac{\mathrm{Cap_n}\left((\mathbb{R}^n \setminus \Omega) \cap \mathrm{B_r}\right)}{\mathrm{Cap_n}(\mathrm{B_r})} > 0. \tag{14}$$

Then either $u \leq 0$ in Ω or there is $\beta > 0$ such that

$$\liminf_{r\to\infty} \frac{\sup_{\{\Omega\cap\{|x|=r\}\}} |u(x)|}{r^\beta} > 0.$$

Note that (14) is **NOT** the Wiener condition at infinity.

Remark 12 *We see that both in the classical result of Example 2 and the new result the Phragmén-Lindelöf yields a fractional growth r^β. However, the condition, "the complement contains a cone" is much restrictive than the capacity condition (14). In fact, there are sets of Lebesgue measure zero which satisfy (14).*

Example 3: This example concerns the Wiener condition at infinity and it is not so well known.

Theorem 13 (Essèn, Haliste, Lewis, Shea, 1985) *Let u be subharmonic in a domain Ω, continuous on $\overline{\Omega}$ and such that $u \leq 0$ on $\partial\Omega$, the boundary of Ω. Assume the Wiener condition holds at infinity, that is,*

$$\int_1^\infty \frac{\mathrm{Cap_n}\left(\mathbb{R}^n \subset \Omega \cap \mathrm{B_r}\right)}{\mathrm{Cap_n}(\mathrm{B_r})} \frac{dr}{r} = \infty. \tag{15}$$

Then either $u \leq 0$ in Ω or u is unbounded in Ω.

THANK YOU FOR YO ATTENTION

Further details in:

On Null Quadrature Domains
by Lavi Karp

To appear in CMFT:
http://www.heldermann.de/CMF/CMF08/CMF081/cmf08006.htm

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